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ON PRACTICAL AND THEORETICAL THINKING AND OTHER FALSE DICHOTOMIES IN MATHEMATICS EDUCATION

Abstract. I owe much of my understanding of the difference between synthetic and analytic thinking in mathematics to my reading of Michael Otte's papers and the conversations we had with him within the BACOMET group. One of the first sources of inspiration for me has been his work on arithmetic and geometric thinking. In the paper I shall outline the consequences of the distinction for analyzing processes of mathematics teaching and learning in my own research. I shall further use this distinction to look critically upon the recent trend in mathematics education of considering mathematics as a kind of "discursive practice."

Key words: epistemological obstacles, manipulatives, mathematics, practical thinking, Pythagoras theorem, teaching, theoretical thinking.

1. INTRODUCTION

This paper, dedicated to Michael Otte, is about practical and theoretical thinking as complementary epistemological categories and the use of this distinction in mathematics education. The distinction is presented as one among many "false dichotomies" that are common in the domain. The dichotomies are first discussed in the light of Michael Otte's papers on complementarity. An alternative view is then proposed in terms of couples of epistemological obstacles. A possible use of the practical/theoretical distinction in mathematics education is illustrated by means of a thought experiment about a teacher educator planning to discuss the use of manipulatives with his student teachers. The thought experiment points to the complexity of the system of objects of thought in mathematics education and the extreme fragility, in practice, of the distinction between theoretical and practical thinking. It also highlights the crucial role that epistemological analyses, such as those offered in Michael Otte's papers, play for research in mathematics education.

In his comments on one of my papers about epistemological obstacles (Sierpinska 1996), Michael Otte was saying that, where I saw a couple of obstacles, he could see only one, namely

> … the problem that in order to understand mathematics one has to take into account [the fact] that mathematics is simultaneously meta-mathematics … [T]he problem lies in an empiricist or concrete epistemology [that] does not think of mathematical objects in relational or structural terms. … [M]athematics is difficult for the learner not because of the technical complications of its method, but because of the specificity of its objects. (Letter dated 24.1.1994)

M. H. G. Hoffmann, J. Lenhard, F. Seeger (Eds.), Activity and Sign – Grounding Mathematics Education, 117 – 135.

He went on to say that he has been busy his entire didactical career with this one problem [the problem of sources of difficulties in mathematics learning] and with the question of the nature of mathematical objects and concepts. In my own research, I have tried to engage more directly with the practice of teaching, with designing and experimenting didactic sequences. Somehow, I always ended up discussing these same problems. They are very powerful attractors indeed in the dynamics of research in mathematics education.

2. EPISTEMOLOGICAL OPPOSITES

Theory of mathematics education is replete with pairs of opposite categories of knowing and thinking such as the empiricist-structural distinction mentioned above, intuition versus formal knowledge, instrumental versus relational, or operational versus structural understanding. In my research I first resisted using such global categories, explaining both the meaning of particular mathematical concepts and students' difficulties by the existence of "epistemological obstacles" specific to concrete mathematical concepts. But, as I went on in my research, I realized (and thus agreed with Michael) that many obstacles were related not to specific concepts but to mathematics in general. And thus I ended up with, first, three categories of thinking in linear algebra: synthetic-geometric, analytic-arithmetic and analytic-structural, and then attributing students' difficulties in linear algebra to their tendency to practical as opposed to theoretical thinking (Sierpinska et al. 1997; Sierpinska 2000; Sierpinska & Nnadozie 2001).

It is tempting to think that these categories refer to some ontological reality; that there exists an identifiable brain activity such as, for example, theoretical thinking, with no trace whatsoever of its opposite, namely practical thinking. But, as Michael Otte has argued in Otte (1990b), these distinctions should be regarded as epistemological, not ontological distinctions. They are our simplified ways of knowing human cognitive activity in mathematics; they are not kinds of human cognitive activity.

> I have argued that, whenever we see mathematical proof as involving only a mechanical aspect, we are driven to see that it involves, as well, an intuitive one. And whenever we are tempted to see mathematical proof as involving only a solitary aspect, we are driven to seeing that it is also a social matter. And whenever we are tempted to see a mathematical argument of the kind found in proof, namely a chain of tautologies or of equalities, as merely, or perhaps the ideal of, literal expression, we are forced to see that it is, in fact, essentially metaphorical. (Otte 1990b)

This is why Otte preferred to speak of "complementarity" (*p* and not *p*) rather than of dichotomy (*p* or not *p*).

3. COMPLEMENTARITY

In his philosophical considerations on mathematics and its teaching, Otte has explored in depth the idea of complementarity of object and method in science (Otte 1990a), or, broadly speaking, the idea that "every scientific explanation simultaneously contains a meta-communication, i. e. it represents, in an exemplary way, an answer to the question what it means to explain an object or a fact at a certain historical point in time." This notion of complementarity comprises issues such as relationships between intuition (which focuses on discovering the object of study) and logic (whose problem is to systematize methods of validation of the findings), immediate perception (synthetic thinking) and discursive procedures (analytic thinking), or between theoretical representation and technology of measurement or computational technique. These issues constitute the philosophical underpinnings of debates on the teaching of mathematics focusing on the problems of striking a balance between "theory" and "practice," knowing why and knowing how, letting the students engage in free explorations and express themselves as they like and teaching them the "right" mathematical discourse and standards of methodological rigor.

Complementarity of these categories could be expressed also in terms of epistemological obstacles. An epistemological obstacle is a way of thinking that stands in the way of another way of thinking, but it would not exist (as an obstacle) without this other way of thinking. Thus it does not make sense to speak of single epistemological obstacles, but only of their pairs. The epistemological categories mentioned above can be seen as pairs of epistemological obstacles in the philosophy of knowledge. Intuition and formal knowledge is such a pair of obstacles, for, without intuition, formalism would have nothing to doubt; there would be no need to formalize in order to confirm or remove the doubt; on the other hand, without formalism, intuition would remain in a state of either permanent self-satisfaction or permanent doubt.

Similarly, theoretical and practical thinking can be viewed as a pair of epistemological obstacles. Thinking is not either theoretical or practical but arises in a tension between the two. The "practical sense" decisions are acts of discarding all but one possible course of action; but this decision would not be necessary if these possible courses of action were not available to the mind. They are available as a result of hypothetical, theoretical thoughts, however primitive, swift and unconscious. On the other hand, the mind would not engage in thinking about the possible courses of action and their outcomes if no action were envisaged at all. As Otte was saying, in his polemic with Piaget's concept of empirical abstraction, which he considered "too primitive" by being completely separated from reflective abstraction (Otte 1990a):

> One has to emphasize that theoretical consciousness demands to conceive the objects and phenomena of reality not just in the form of knowledge and contemplation but as parts of activity also ... [T]he relationship between the conceptual-reflective and the algorithmic-logical elements of mental activity is only conceivable as an interaction of two poles of a relationship the basis of which is the activity. (Otte 1990a)

"This is all very well" – a mathematics teacher might say at this point – "but what difference does it make for my teaching practice, whether I see these pairs of categories as dichotomies or as complementary couples?"

4. THE QUESTION OF RELEVANCE OF EPISTEMOLOGICAL DISTINCTIONS FOR MATHEMATICS EDUCATION

Saying that mathematical thinking is, at the same time, intuitive, formal, practical, and theoretical is anything but an astounding discovery. Of course, the realization of the epistemological complementarity discussed above might save one the inevitable failure of organizing one's teaching on the basis of the assumption that, say, "real" mathematical thinking is formal and theoretical, and that the intuitive and practical aspects of knowledge construction are only the necessary contingency of some shameful "didactic transposition" from scholarly research knowledge to the social and cultural institution of teaching mathematics to masses of students. It might save one, as well, from trying to "derive" theoretical concepts from concrete, "hands-on" experience, based on the belief that the meaning of these concepts is somehow already there in the empirical relations. Steinbring has convincingly demonstrated the ineffectiveness of such approaches, using the theory of epistemological triangle and detailed analyses of classroom interactions (e. g. Steinbring 1991, 1993).

But the mere realization of complementarity cannot help mathematics educators in understanding what exactly is difficult in learning this or that mathematical idea in a particular teaching/learning situation, never mind helping them in planning and organizing such situations. In each case, the mathematics educator must "roll up his sleeves" and do the epistemological and didactic analysis almost from scratch. This is no trivial task, as can be seen from the above-mentioned papers by Steinbring.

The mathematics educator must also be more specific in describing the epistemological categories if he[1] intends to use them in analyzing particular teaching situations; he needs to "operationalize" them. With respect to the theoretical/ practical distinction, for example, saying that learning mathematics is difficult because it requires theoretical thinking is almost a tautology. In the next section I present a characterization of the theoretical/practical distinction, which we developed for the purposes of our research on linear algebra teaching and learning (Sierpinska et al. 2002).

5. A CHARACTERIZATION OF THEORETICAL/PRACTICAL THINKING

Michael Otte once told me that the difference between synthetic and analytic thinking is that the former holds a direct relationship with its object while in the latter this relationship is mediated by one or more sign systems. The same can be said of the difference between practical and theoretical thinking, since it is normally assumed that theoretical thinking is analytical.

Thus what is theoretical or practical is not thinking as such but *the relationship between thinking and its object.* It makes sense to conceive of this object as some kind of action, actual or imagined, present or past, performed or planned, since both theory and practice are normally related to purposeful action. "Action," here, could mean proving a mathematical statement as well as carving squares from a plank of wood. Practical thinking could be viewed as *thinking-in-action*, whereby changes in thought directly influence changes in action. Thus, the relationship between thinking and this very action of thinking is practical. If a philosopher ponders a theoretical question, the relation of his thinking to this very activity of thinking is necessarily practical; he is not thinking about his thinking. He is just thinking; he is engaged in the practice of philosophizing.

For theoretical thinking to even begin, the thought and its object must belong to different planes of action (Figure 1). Thinking-*in*-action must become thinking-*about*-action. The moment the philosopher reflects back on his thinking, verifying if it is well founded, eoretical.

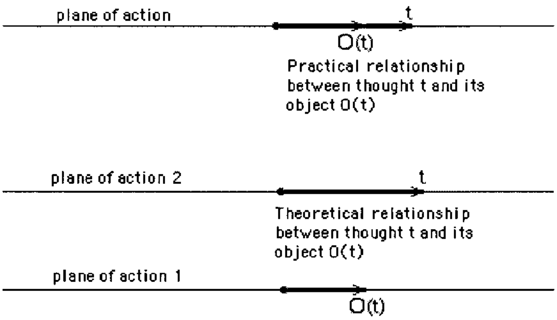


Figure 1. Relations between thought and its object in theoretical and practical thinking.

Let me illustrate this idea with one more example. Imagine a student who solves an equation and then substitutes the obtained result into the original equation. In the phase of solving the equation, the student is engaged in the practice of solving equations: his thinking and his activity of processing the algebraic expression belong to the same plane of activity. In the phase of substitution, the student may be taking a step back from his previous activity, which would now become the object of his thinking. He may be verifying if the result he obtained indeed satisfies the equation. In this case, we could say that the student is "engaged in theoretical thinking" or, more precisely, that the relationship between his thinking and its object is theoretical. But the student may also do the substitution as part of what he understands as the school task of "solving an equation," without viewing it as a means of control of the result obtained in the first phase. It is well known that many students indeed hold this conception and are not bothered if they obtain a contradiction through substitution. These students think practically in both phases of the task.

Obviously, one cannot assume that belonging to different planes of action is a sufficient condition for the relation between thought and its object to be theoretical. Musing about days gone by, day-dreaming, or rotating three-dimensional shapes in one's mind would then count as theoretical thinking and this is not what we intend to mean. More restrictions on the relationship between thought and its object are needed for a satisfactory characterization of theoretical thinking.

The most obvious characteristic of what we normally call *theoretical* thinking is that its ultimate purpose is the production of theories or conceptual systems.

One consequence of this assumption is that theoretical thinking is not about techniques or procedures for well-defined actions, although these might be derived from or explained by the theories. Theoretical thinking is *reflective* in that it does not take such techniques or procedures for granted but considers them always open to questioning and change. In this sense, therefore, theoretical thinking is opposed to mythical thinking, in which knowledge is considered as "natural" or "sacred" and therefore in no need for justification (Steinbring 1991).

Another consequence is that theoretical thinking is *systemic*, i. e. its objects are not particular actions but systems of relations between actions, and systems of relations between these relations. As Otte was saying,

> The history of science may be briefly sketched as a transition from thinking about objects to relational thinking. Theoretical thinking, accordingly, is not concerned with concrete objects, nor with intrinsic properties of such objects, and theoretical terms, in particular, are not just names of objects. Rather, science is concerned with the relationships between objects or phenomena. As the historical transition took place, it became increasingly obvious that a theoretical term will receive its solid content, its clear form, only from its relationship to other concepts. (Otte 1990a)

The systemic character of theoretical thinking entails *sensitivity to contradictions*; otherwise, conceptual systems would collapse. Vygotsky has particularly stressed this characteristic of scientific, as opposed to everyday concepts (Vygotsky 1987, 234). Actually, the very concept of contradiction makes no sense outside a system of concepts. Contradiction is a type of logical relationship between propositions; there can be no contradiction between events occurring in space and time; their meanings change with the context in which take place. Contradiction thus requires *stability of meanings* in the frame of reasoning. This can be achieved by definitions and other agreed upon characterizations.

The combination of reflective and systemic thinking implies that theories do not grow by simple addition of new concepts, but that new developments may cause a restructuring of the whole system. The system is always reflected upon as a whole. This feature of theoretical thinking is sometimes called "*reflexivity*" (Steinbring 1991).

Concern with non-contradiction implies that attention is being paid to problems of *validation*, both at the level of the systems themselves and at the meta-level, i. e. at the level of *methodology*. Theoretical thinking asks not only, *Is this statement true?* but also *What is the validity of our methods of verifying that it is true?* Thus theoretical thinking always takes a distance towards its own results.

Thinking within conceptual systems can only produce conditional truths; it is *hypothetical thinking*. Theoretical thinking is concerned with problems of the sufficient, necessary, essential, complete character of conditions of truth in each case.

As mentioned, the assumption of belonging to different planes of action already implies that theoretical relationship between thought and its object is *analytic*, i. e. mediated by systems of signs. But, if we assume that the results of theoretical thinking are conceptual systems or theories, which have to be formulated in some coherent terminology and symbolic notation, then we must also require that theoretical thinking have an analytic relationship with sign systems themselves. Theoretical thinking not only is mediated by systems of signs; it takes systems of signs as an object of reflection and invention.

In brief, theoretical thinking is thinking where thought and its object belong to distinct planes of action, and whose purpose is the production of internally coherent conceptual systems, based on specially created systems of signs. Theoretical thinking is, therefore, reflective, systemic and analytic.

I have argued elsewhere (Sierpinska et al. 2002) how highly relevant, a priori, are the above features of theoretical thinking in understanding linear algebra, and how irrelevant they can be for high achievement in linear algebra courses. In this paper, I will focus on the complementarity between theoretical and practical thinking in actions related to teaching and learning of mathematics.

6. A THOUGHT EXPERIMENT: THE INTERPLAY OF THEORETICAL AND PRACTICAL THINKING IN A TEACHER EDUCATOR'S PLANNING OF AN ACTIVITY ON THE USE OF MANIPULATIVES WITH STUDENT-TEACHERS

Thinking in and about mathematics education involves simultaneously several planes of action. In particular, the object of study for a mathematics education researcher may comprise several levels of recursion of the act of "theoretical reflection on practice." For example, when a researcher reflects theoretically on the practice of a teacher educator, he may use his practical experience of being a teacher educator, a schoolteacher, a learner and doer of mathematics, and a researcher knowledgeable of the theories and methodologies of his field. He may entertain, with each of these planes of action, a practical or a theoretical relationship.

In any concrete activity of reflection, these relationships are closely intertwined and dependent on each other. Their identification and categorization is possible in a methodological analysis, but not in actual fact. This is what I would like to show in the following thought experiment.

Suppose a researcher reflects on the work of a teacher educator preparing an activity for his student teachers aimed at a reflection on the use of manipulatives in mathematics teaching, on the example of the learning, by high school students, the meaning of the Pythagorean theorem. In the first section (6.1), the narrator is the hypothetical teacher educator. In the second (6.2), a researcher interprets and analyzes the actions of the educator, focusing on the interplay between his theoretical and practical thinking.

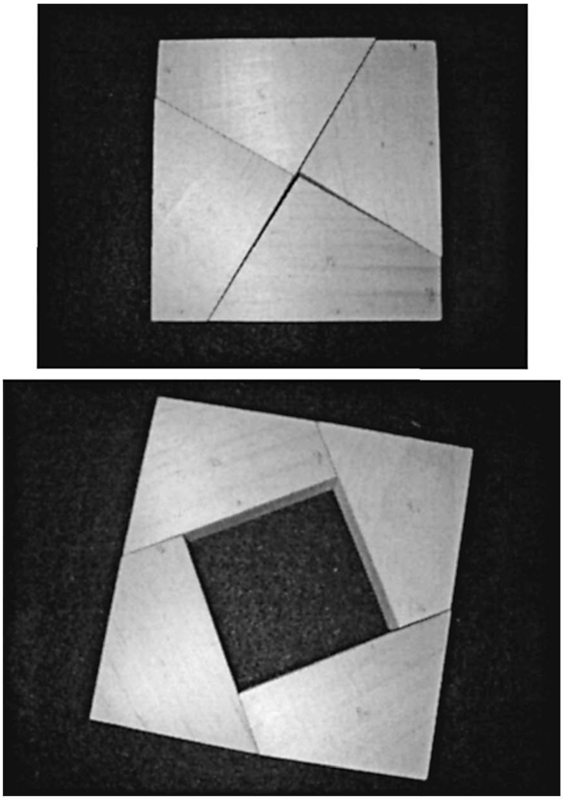
6.1 Teacher educator prepares a class on the use of manipulatives

[1] Suppose I am a teacher educator preparing a session with student teachers on the problem of using manipulatives in the teaching of mathematics. I want to convince them that mathematics is not there, in the manipulatives, but, at best, in the interplay between the practical and theoretical tasks based on actions with the manipulatives.

[2] Let me prepare for a worst-case scenario. Suppose the pre-service teachers in my class want a straightforward judgment such as, "manipulatives are good" (or bad). Also, suppose they expect that teaching with manipulatives is easy:

one just goes into the classroom with a bag of manipulatives, lets the students play with the them, and the students thus "naturally" discover the mathematical concept planned for this particular lesson.

[3] What situation could help my students realize that there is no simple recipe and that it all depends on the manipulatives, what you want to teach with them and how you set up the didactic situation? I know that just telling teachers "it depends" will not help them understand the complexity of the issue. I need to engage them in planning a concrete lesson with concrete manipulatives. Suppose I take the wooden puzzle that I got at the last NCTM[2] meeting and ask student teachers to imagine if and how they could use it to introduce the Pythagorean theorem.



Figures 2a & 2b. Two ways of arranging the pieces of the puzzle.

[4] This, I feel, is bound to show them that while manipulatives may embody mathematical ideas for those who already have them in their minds, they are not necessarily helpful in bringing these ideas to the minds of those who hadn't seen them before.

[5] The puzzle has 4 pieces, which can be arranged into shapes like those in Figure 2.

[6] This pair of shapes brings to mind the "puzzled" proof of the Pythagorean theorem, as in Figure 3.

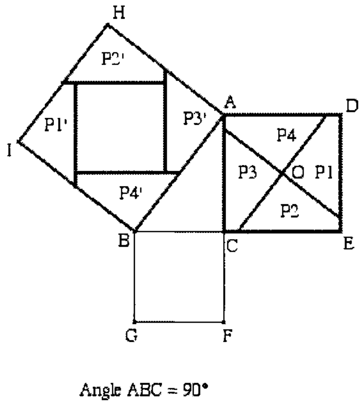


Figure 3: The idea of the popular "puzzled" proof of the Pythagorean theorem.

[7] The student teachers will probably recognize the Pythagorean theorem in the puzzle, and they will take it for granted that their students will "see" it as well, in spite of never having heard of the theorem before.

[8] I will show them that this need not necessarily be so. I will invite them to imagine, step by step, what may happen if they bring the puzzle to the classroom and ask the students to first play with it freely and then to construct squares. I will ask them to assume that students in the classroom are mostly practically minded. I don't know what scenarios they may come up with, but let me do this exercise myself, so I can be better prepared for arguing with their claims.

[9] Most students want to make nice looking material objects. They do not think of a shape first and then try to construct it, but just move the pieces around, trying in which ways they best "fit" with each other. Their decisions about when to stop and consider the shape done are based on visual and tactile clues and their spontaneous esthetic feelings. These may be explained by their previous encounters with cultural artifacts, but not by some explicit esthetic principles such as "symmetry," "compactness," or "balance" (Figure 4).

[10] If the students only want to play with the puzzle in this rather random fashion, they will never be brought anywhere close to the Pythagorean theorem. Let me now think of the next-to-worst scenario. The students start noticing some relations between the pieces. They might discover that the four pieces of the

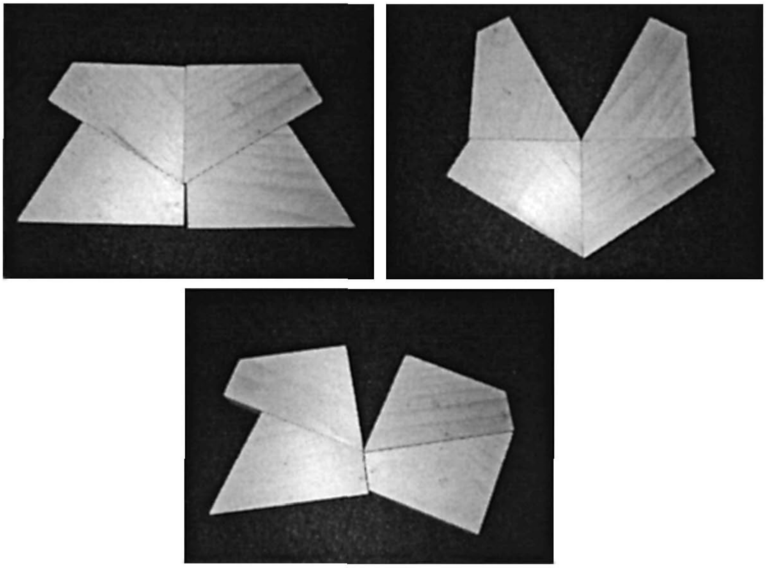


Figure 4. Shapes that could be obtained by students through free play with the puzzle.[3]

puzzle are not identical. The lengths of their sides differ a little. Especially one piece is quite off the shape of the other three. Also the angles that look like right angles are not exactly so, because, when the pieces are put side by side, they do not form a straight line exactly (Figure 5).

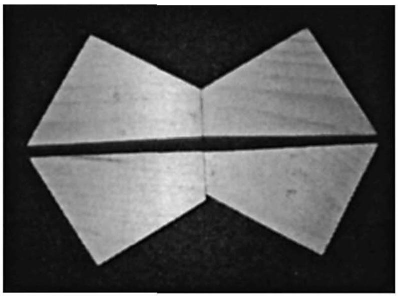
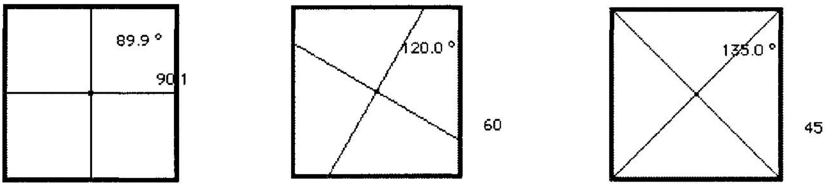


Figure 5. The pieces of the puzzle are not all identical.

[11] Students may decide to ignore the differences (as technical errors of the person who cut the pieces). Suppose now that some students are technically minded or have been inspired by their recent experiences in the woodwork class. Some of them may start thinking about the technology of producing the puzzle. This may lead them to viewing each puzzle as made from a single square piece of wood (like in Figure 2a) cut along two perpendicular lines passing through the center of the square, constructed as the intersection of its diagonals. Some stu-

dents may measure the angles at which the inner segments fall on the sides of the square, find that they are approximately 60° and 120°, and include these measures in their definition of the puzzle. Other students may see these angles as arbitrary and only constrained by the requirement of producing a non-trivial puzzle, i. e. one made of four quadrilaterals with unequal sides and not four squares or four right-angled triangles (Figure 6).



Figures 6a, 6b, 6c. Two trivial and one non-trivial puzzle.

[12] These technological concerns of the students could perhaps be considered as the most "natural" outcome of playing with the puzzle in a high school mathematics class. This situation could give the teacher an opportunity to generalize the puzzle as a set of four identical quadrilaterals with two opposite right angles and the sides of one of the right angles[4] being equal. The other two angles add up to 180°, because the sum of angles in a convex quadrilateral is equal to 360°. Thus, if one of the angles measures a, the other measures 180° – a. If a = 90°, the piece is a square[5] (Figure 6a); if a = 135° the piece is a triangle (Figure 6c).

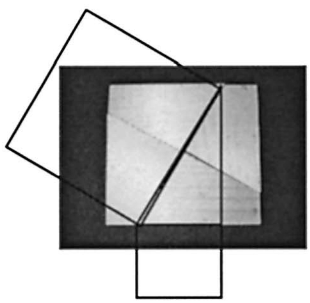


Figure 7. The three squares seen as built on the sides of a right angled triangle.

[13] The question is: Is it at all possible to bring students to think about the Pythagorean relation from playing with the puzzle? Is there a best-case scenario? Suppose the students construct the squares in Figures 2, either by themselves or in response to the teacher's explicitly formulated task. Suppose they even notice that there are three squares in these two figures and that the area of the external square in Figure 2b is equal to the sum of the square in Figure 2a plus

the area of the square built on a segment which can be seen as a certain part of the side of the square in 2a. Suppose, even more optimistically, that they notice that all three squares can then been seen as built on the sides of a right-angled triangle (Figure 7).

[14] Making these observations requires that the students be highly theoretically oriented. It requires seeing the shapes obtained with the puzzle as structures composed of segments of different lengths and mutual positions. It also requires reflecting about the relations between the different shapes obtained with the puzzle (possibly in a situation where only one shape is available to the senses at a time). These observations are not a result of direct visual and tactile perception: they are a result of a construction of a geometrical model of the puzzle (Figure 8).

[15] Communication of these observations among students would require coding the different segments of the pieces of the puzzle. Students would not know where their observations would be leading them, so they would be likely to use ad hoc representations, such as color. However, color is not functional if algebra is to be used later on in the representation of the Pythagorean relation and its proof. If the teacher imposes a notation, this will immediately destroy the "naturalness" of the situation. The students will know that their initiative does not count and this is not real exploration but the well known ritual of fake "discovery teaching," where students are left in the dark till they are eventually explicitly told what they were expected to have discovered. But suppose that somehow students are brought to using letters to denote lengths of segments, as in Figure 8.

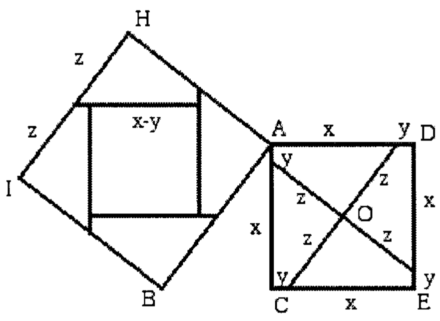


Figure 8. Using a diagram to compare the sides of the three squares.

[16] The students would be probably quick to notice, but also take it for granted that, in the left-hand side square in Figure 8, the side of the external square is 2z and the side of the internal square is $x - y$. It could also be obvious for them from the figure that the side of the right-hand side square is $x + y$. Using the known formula for the area of a square, the students might write the relation: $(2z)^2 = (x + y)^2 + (x - y)^2$. The students would now see in front of them a famil-

iar mathematical object: an algebraic expression. Their aim might become to simplify the expression (to $2z^2 = x^2 + y^2$). This is what they have always done in such situations.

[17] The teacher would not be satisfied with this result. Not because it is not true. It is, but it is also irrelevant from the point of view of his didactic goal. The relation could be obtained directly from looking at the square ACED, by noticing that $4(xy/2 + z^2/2) = (x + y)^2$. Taking $y = 0$, it could lead to the formula for the diagonal of a square ($2z = \sqrt{2}\, x$), which, in the curriculum, is only derived as a consequence of the Pythagorean theorem.

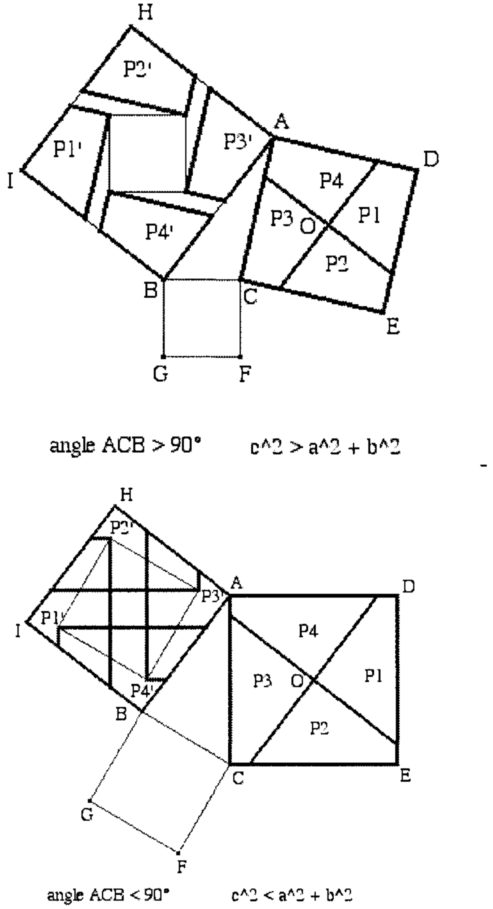


Figure 9. The failure of the Pythagorean identity $c^2 = a^2 + b^2$.

[18] At this point, the student teachers should be convinced that, in order to even start discussing relations among the three squares that can be obtained with the puzzle, students have to forget about the puzzle as a puzzle altogether. They should also realize that students would have to be heavily directed to focus

their attention on the relation between the areas of the squares without simplifying it any further. But even if the hypothetical teacher achieves all that, his students will still be extremely far from the "discovery" of the Pythagorean theorem. This is because, for the puzzle, the relation between the areas is always true. In fact, it is quite obvious and trivial. But the Pythagorean theorem speaks about one very exceptional situation. The rigid wooden puzzle illustrates this one single exceptional situation without hinting at the class of situations of which it is an exception. It is, indeed, quite exceptional that the areas of squares built on two sides of a triangle add up to the area of the square built on the third. It only happens when one of the angles of the triangle is a right angle. The puzzle, in itself, is unable to provoke students to think about the conditions of the Pythagorean relationship between the areas of squares built on the sides of a triangle. At best it illustrates a possible way of proving the theorem once it is realized as a conjecture.

[19] But the student teachers should not be left with the impression that the only way to introduce the Pythagorean theorem is to state it on the board and have the students learn it by heart. They have to understand that their students will not appreciate the significance of the theorem this way, either. Suppose I suggest that student teachers try to imagine starting a lesson by directly asking the theoretical question: What is the relation between the areas of squares built on the sides of a triangle? and allowing their students to work within a dynamic computer environment (Figure 9).

The discussion would then be organized on their views of the potential of this type of more sophisticated "manipulatives" in the teaching of the Pythagorean theorem.

6.2 Analysis of the thought experiment

This section presents a possible theoretical reflection of a researcher on the role of theoretical thinking in the work of a teacher educator planning a teaching activity with student teachers. The analysis will make references to the narrative of the hypothetical educator in the form of paragraph numbers in square brackets. It will also make explicit the evaluation, as theoretical (t) or practical (p), of the narrator's thinking about the practices of research (R), teacher education (E), teaching (T), learning (L), doing mathematics (M). The analysis highlights in italics words that are related to particular features of theoretical thinking.

In [1] the researcher engages in hypothetical thinking ("suppose") about the action of a teacher educator, so his relationship with E is theoretical (tE). The choice of the topic, however, is based on his experience with E; he knows that manipulatives is a "hot issue" and is likely to attract student teachers' attention (pE). He also knows that this is a controversial issue in mathematics education (pR) and has a theory about the epistemological relationship between manipulatives and mathematics (tM). This theory re-surfaces now and again in his reflection ([4], [7], [8], [10], [13], [14], [18]).

In [2] it is the hypothetical teacher educator who speaks. His consciously adopted methodology of preparing a class (tT) is to first "prepare for the worst" and then gradually consider more optimistic scenarios. His worst-case scenario is based on the assumption, founded on his experience with teaching (pT), that his actual student teachers as well as hypothetical teachers and pupils have a strongly practical attitude towards their tasks. He considers this to be the worst-case scenario, because he assumes that mathematics is theoretical knowledge par excellence (tM).

In [3] the educator decides against just telling the student teachers that the use of manipulatives can be more or less effective depending on circumstances. Following perhaps a "socio-constructivist approach" (tT), he plans to confront his student teachers with a specially designed situation, let them draw the conclusions for themselves, and then engage in an argument with them, negotiating alternative ways of thinking. This operationalization of the socio-constructivist epistemology in terms of didactic choices is based on his familiarity with its common interpretations within the community of teacher educators to which he belongs (pE). He chooses to use a wooden puzzle as an example of a manipulative, because it is there on his desk, reminding him of his recent activities with children of various ages playing with the puzzle (pT).

In [5] the educator reflects on (tM) his personal experience with the puzzle; he has played with the puzzle, trying to make mathematically meaningful shapes (pM). Based on this experience, he assumes, in [6], that knowing the Pythagorean theorem allows one to construct a material model of the idea of the proof of the theorem with the puzzle (pM, tM).

In [7] the educator reasons as follows: Since the student-teachers know the Pythagorean theorem, and, according to the worst-case scenario, they hold the naïve belief that mathematical patterns are there in nature and things (tL), waiting to be discovered, it is very likely that they will expect high school students to "discover" the theorem through playing with the puzzle (tT).

In [8] the educator reflects on the possible moves (tT) in this situation, based on his experience as a teacher (pT). The best thing would be to ask the student teachers to actually perform an experiment with a student who has never seen the Pythagoras theorem before. But the constraints of time as well as the practical difficulties of access to such students and of the control of the experiment by the teacher educator make him opt for a collective "thought experiment" instead.

In the sequel of the thought experiment ([9]-[18]), the teacher educator speculates about how his students could be led to the realization of the non-transparency of manipulatives by imagining what could happen in a classroom started by a free play with the puzzle.

The educator imagines the course of events based, again, on his methodology of going from the worst-case scenario to gradually more optimistic scenarios regarding the agents' theoretical thinking (tT, [9], [10]). His speculations are founded on his informal observations of students playing with the puzzle (pT, tL, [9], [10]), his theory of people's relationship to cultural artifacts (tL, [9], [10]), and his reflection on his experience of mathematizing the relationships between the elements of the puzzle (pM, tM, [10], [11], [12]).

In [12] he reflects on the outcome of these speculations (tT); he considers a technical approach to the puzzle as quite natural in students. On the other hand, thinking about the Pythagorean relation in the context of the puzzle does not appear as natural; based on his reflection on his activity of relating the puzzle configurations with the Pythagorean relation, he realizes that this would require noticing unobvious quantitative relations and highly theoretical thinking (pM, tM, [13], [14]). This would also require a graphical representation of two special configurations of the puzzle and a mathematically consistent coding of the elements of the puzzle (pM, tM, [15]).

Para	tR	pR	tE	pE	tT	pT	tL	pL	tM	pM
1		1	1	1					1	
2					1	1			1	
3				1	1	1				
4									1	
5									1	1
6									1	1
7					1		1		1	
8					1	1			1	
9					1	1	1			
10					1	1	1		1	1
11									1	1
12					1				1	1
13					0				2	1
14									2	1
15							1		1	1
16					1		1		1	
17					1		1		1	
18					1		1		2	
19					1				1	1
Sum:	0	1	1	2	11	5	7	0	20	9

Figure 10. A summary of the interplay of theoretical and practical thinking in the course of the educator's work of preparing his classroom activity. Theoretical thinking was invested mostly into the educator's relationship with the practice of doing mathematics and the practice of teaching. This thinking was strongly supported by the educator's experiences in these domains of practice. His thinking about learning was more speculative.

These reflections lead the educator to point to the shaky foundations of the so-called "discovery learning" (tL, [15]). Through [16]-[18] he demonstrates (tT, tL, tM) how unrealistic it is to expect that the puzzle will "naturally" lead students to thinking about the Pythagorean theorem, in all these scenarios, not only in the worst case scenario, based on a reflection on his own mathematization of the puzzle. He shows that even if students are highly theoretically minded, the puzzle cannot bring them to thinking about the Pythagorean theorem, if they hadn't seen it before, because the theorem points to the conditions of existence of a puzzle such as the given one. This existence is not put into question in the puzzle; the puzzle is a fact.

The educator eventually goes back to thinking about the possible reactions of his student teachers to the realization of the epistemological impossibility of obtaining the Pythagorean theorem through even theoretical modeling of the puzzle. Based, again, on his methodology of worst-case scenario, he prepares to counter the probable student teachers' conclusion that "manipulatives are bad" with a proposal of an alternative representation (tT, pM, tM, [19]).

Figure 10 contains a summary of the above analysis of the educator's engagement with the different domains of practice.

A striking overall characteristic of the teacher educator's thinking is the lack of one coherent theoretical framework or conceptual system, on which his planning would be based. The educator makes decisions based on bits of various "theories," while being strongly influenced by his own experience and practice of teaching, learning and doing mathematics. His relationship to the different objects of his reflection can be regarded as locally, but not globally theoretical. He makes conscious use of a methodology, but does not reflect on its validity. He does not verify for contradictions among his conclusions drawn at different points in his planning. His aim is to produce a rich learning experience for his student teachers, not to construct a theory of the use of manipulatives in the teaching of mathematics.

The next section contains a theoretical reflection of the researcher on the results of this thought experiment and, more generally, on research in mathematics education (tR).

7. CONCLUSION

It was not too difficult to write a characterization of theoretical as opposed to practical thinking. Innumerable philosophers did that, at least from the time of Aristotle. It was much harder to use this distinction in speaking about a concrete instance of thinking about teaching, learning and doing mathematics. One reason for this difficulty is the complementarity of the categories of practical and theoretical thinking. Both are related to action, one engaged with action from within, the other – from without. This is a subtle difference and it is easy for the researcher to mistake one for the other.

At any given moment, the thinking subject is involved in a practical relationship with an action, planning what to do next. But any decision that is being made in the course of this action depends on a consideration, however swift, of the hypothetical possibilities and the choice of one. The choice may be based on various degrees of theoretical analysis and construction. It is not possible to reliably judge such momentary choices as based or not on theoretical thinking – and this is another source of the difficulty. One can only speak of the presence of perhaps some features of this kind of thinking and one can never be sure if this short instance of thinking was done with some global and conscious intention of theory construction. "Intention" and especially "conscious intention" are categories that have caused enough problems in philosophy and psychology; it is very difficult to operationalize them in research.

Another reason of the difficulty is the complexity of what goes on in people's minds. Thinking takes place simultaneously at several planes of action, which can be considered separately only in theory, and even then, hypothesizing about the thinking at all of these planes in a subject at a given moment of an observation or interview may easily overwhelm even the most assiduous of researchers. This complexity cannot be ignored in mathematics education research, because its object is exactly the interplay of thinking at several levels of action at once. The construction of a coherent theoretical framework for the object of research in mathematics education is, therefore, an extremely challenging task (but not an impossible task; see, e. g. Brousseau 1996; Chevallard 1999).

It is not surprising, therefore, that so many researchers in mathematics education tend to reduce the complexity in their work, and either use eclectic approaches or focus on some chosen planes of action. Certainly cognitive and socio-cognitive issues, and philosophical questions related to the nature of mathematics have attracted much attention.

One is often tempted to deplore this state of affairs. However, as in the thought experiment described in this paper, the crucial argument in analyzing a teaching project is often found not by applying the most general and sophisticated theoretical framework, but by looking at the best-case scenario. Of course, if students are not interested or not intellectually mature for a topic, and the teacher makes pedagogical mistakes, the project will fail. But suppose students are capable and willing to think theoretically about mathematics, and the teacher is "good" according to the standards of some accepted instructional theory. If a teaching approach does not fulfill the expectations in this situation, the reason is not in the pedagogy but in the epistemology of the subject matter. Epistemological analyses of the mathematical ideas are, therefore, the foundation of any teaching project in mathematics education. This is why the work of philosophers such as Michael Otte is so important for our domain.

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NOTES

[1] The pronoun "he" is used throughout the paper as a generic pronoun, not as this author's political statement.
[2] Regional conference of the National Council of Teachers of Mathematics, Montreal, Canada, August 2002.
[3] These shapes were the first three produced by a 6:9 years old girl after she was given the puzzle and asked to "make some shapes with it." Asked why she made the first shape just so, she answered, "because it fitted with those triangles. And it also looks a bit like a flower, like those you get in a computer." The second shape was described as "it looks like a funny cat;" about the third she said, "it's a butterfly that's acting weird." Two grade seven students, asked to play with the puzzle, spontaneously constructed similar shapes. They were mainly interested in verifying how the parts of the pieces fitted with each other; e. g., if it was possible to make a straight line with two of them. These students were able to construct the

two squares in approximately 10 minutes. One of them believed that the inner square (in Figure 2b) is of equal size with the full square (in Figure 2a).

[4] Here, the meaning of the word "angle" may be based on an intuitive/visual idea of "corner."

[5] At this point, "square" means a quadrilateral with 4 right angles and equal sides, not a visually grasped regular shape.

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