MARIA G. BARTOLINI BUSSI, MARA BONI, FRANCA FERRI and ROSSELLA GARUTI

EARLY APPROACH TO THEORETICAL THINKING: GEARS IN PRIMARY SCHOOL

ABSTRACT. Gears are part of everyday experience from very early childhood. This paper analyses a teaching experiment conducted with 4th graders in the field of experience of gears. The aim is to identify the characteristics which, given a suitable sequence of tasks and proper teacher guidance, have enabled the pupils to approach theoretical thinking, and in particular mathematical theorems. We have focused on the relationships between the epistemological analysis of some pieces of mathematical knowledge brought into play in tasks concerning gears, cognitive analysis of pupil construction of those pieces of mathematical knowledge, and didactic analysis of the teacher's role in designing tasks and in offering cultural mediation. This paper presents the early findings of the teaching experiments, both at the external level of interpersonal classroom processes and at the inner level of individual mental processes.

KEY WORDS: gears, mathematical discussion, primary school, semiotic mediation, theorems, theoretical thinking

1. INTRODUCTION

The theoretical framework of this research study follows the Vygotskian trend, with emphasis on the social construction of knowledge and on semiotic mediation by means of cultural artefacts: the social dimension is embodied in recourse to 'mathematical discussion', orchestrated by the teacher (Section 2, Bartolini Bussi,1996); cultural artefacts are represented by gears, figures and theories (i.e. Euclid's geometry and Heron's kinematics) by means of which mathematical modelling of activities with gears is accomplished. From the outset concrete referents from everyday life are used in the classroom. This choice recalls the seminal work done from the sixties in Italy by Emma Castelnuovo (Castelnuovo and Barra, 1980) and is consistent with the studies carried out in the trends of 'realistic mathematics' (Freudenthal, 1983; Treffers, 1987) and 'operative concept formation' (Bender and Schreiber, 1980). This paper analyses a teaching experiment conducted with 4th graders in the 'field of experience' (Section 2) of gears. The aim is to identify the characteristics which, given a suitable sequence

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of tasks and proper teacher guidance, have enabled the pupils to approach theoretical thinking, and in particular 'mathematical theorems' (Section 2). The experimental part of the study is an example of research for innovation (Bartolini Bussi, 1994), in which action in the classroom is both a means and a result of progressive knowledge of classroom processes (Section 3). In Section 4 we shall present the chronicle of a two-year long teaching experiment. The data discussed in this paper come from a 4th grade class, and are consistent with the data obtained in the ten other classes involved in the experiments (which ranged from 1st to 6th grade). We shall show that most of the pupils have succeeded in producing general, abstract and conditional statements (Boero and Garuti, 1994) about the motion of gears, and are able to construct different kinds of argumentation to defend them. At the same time, they have been involved in the social construction of a theory (namely a piece of Heron's kinematics) and have worked with scripts of mathematical proofs. In Section 5 we shall shift to the level of individual processes and discuss two crucial (and intertwining) issues: the genesis of signs (e.g. arrows) in problems concerning the motion of gears; and the genesis of early theorems about the motion of gears. In the final discussion (Section 6) we shall relate the findings of this experiment to the current literature on the topic.

2. Theoretical framework: An outline

Gears are elements of a field of experience. The term 'field of experience' is used after Boero et al. (1995) to mean 'the system of three evolutive components (external context, student internal context, teacher internal context), referred to a sector of human culture which the teacher and students can recognise and consider as unitary and homogeneous'. The field of experience evolves over time through the social practices of the classroom. In this experiment, verbal classroom interaction takes place by means of 'mathematical discussion, i.e. a polyphony of articulated voices on a mathematical object, which is one of the objects-motives of the teaching-learning activity' (Bartolini Bussi, 1996). The polyphony of voices concerns: the dialogue between the voice of everyday practices, evoked by concrete referents, and the voice of mathematical theories; the dialogue between complementary theories (i.e. Euclid's geometry and Heron's kinematics), which are useful for modelling experiences with gears when different foci are assumed.

2.1. *The external context*

The external context is determined by the concrete objects of the activity (gears – e.g. toothed wheels, chains, racks and endless screws; and signs – e.g. gestures, figures, texts). From the outset of activity, the field of experience is furnished with geared mechanisms that are a common part of everyday life (e.g. kitchen tools like the tin opener, vegetable drier, corkscrew and egg beater; bicycles; odometers; mills; clocks; toys). These all have a physical corporeity with constraints and objective relationships that determine the ways they can be used. The field of experience evolves by means of the concrete practices carried out (manipulations, tasks of representation, interpretation, prediction, mathematical discussions and so on). The time span for the development of the external context (which requires months or even years) is that of the didactic system (Brousseau and Centeno, 1991), where long-term evolution in the senses of the objects of knowledge can be planned and observed (Section 4).

2.2. *The internal context of the pupil*

The internal context of the pupils is influenced by the quality and intensity of exploration of the external context, as performed in both out-of-school experience and in classroom tasks under the teacher's guidance. This context develops over time. Let us first consider everyday situations: in a kitchen a child and an adult are beating eggs to make a cake; in a courtyard a child and an adult are repairing a bicycle cog. If the adult involves the child and discusses the activity to be performed, the unity of perception, speech and action which produces the inner visual field (Vygotskij, 1978) is realised. Not all children are so lucky to have this type of out-of-school experience: more often they can only observe from a distance or act without stimulating interlocutors. For these pupils in particular it is important to introduce guided and reflective concrete practices in the classroom and to create a dialogue between this practical experience and the theoretical experience of modelling by mathematical tools. To describe and analyse the individual short-term mental processes in the solution of specific tasks, we need to consider time as an inner variable too (Boero and Scali, 1996). Inner time is governed by its own rules: for instance it can be reversed, contracted, enlarged, cut and displaced – in the past or in the future – according to the specific goal of the subject (e.g. remembering, planning, foreseeing) in order to accomplish dynamic mental experiments. The mental processes of the learners may be accessed by interpreting the external traces (signs, metaphors, gestures) that can be observed (Section 5).

2.3. *The internal context of the teacher*

The internal context of the teacher (and the researcher as well) is influenced by the quality and intensity of exploration of the external context, as performed by the research team or as a personal project. The specific research methodology used in the project (Section 3), requiring the active involvement of teachers in all phases, makes this experiment an interesting laboratory for studying the evolution of the teacher's internal context. We have not focused on this aspect as a research issue.

2.4. *The reference culture: Theories and practices*

In physical experience with gears, there is perception of both the shapes of objects and the final end determined by the chain of motions. The two aspects can only be separated artificially for the purpose of modelling. If the aim is to model shapes (for instance to produce or to interpret a technical drawing representing a real or designed gear), the Euclidean plane or space geometry come into play. An example of a planar gear is given by a roller corrector (see Figure 1), while an example of a spatial gear is given by a mill. The former can be represented by means of geometrical constructions with straightedge and compasses, whilst the latter calls for technical drawing tools. If motion (i.e. functioning) is in the foreground, static geometry is not sufficient: at very least a kinematic approach is needed. In this paper we shall analyse only the directions of rotation, which, in the modern approach, are represented by arrows (see Section 5.1). A possible reference theory is clearly stated by Heron in Book 1 of his Mechanics (1st century AD): 'Two circles in gear by means of teeth turn in opposite directions. One turns right, the other turns left' (Carra de Vaux, 1988). Drawing on this single postulate, several interesting theorems are stated and proved. For instance: 'If a third wheel is in gear with each of two wheels in gear, the gear does not work'. In the following, this 'clover' configuration will be referred to as 'the elementary block'. With n wheels in gear as in a necklace, the gear works if and only if n is even. The clover configuration is the paradigmatic example that emphasises the need for both theories: in the Euclidean framework, the problem of drawing a third circle that touches two tangent circles can be solved for any given radius; yet, if the problem is to construct a working gear, this solution is not acceptable (Ferri et al., 1997).

The reference culture enters the classroom by the teacher's mediation. According to Vygotskij (1990) scientific concepts are always included in a coherent conceptual system, that is connected with a cultural tradition, hence they are acquired through instruction: 'Scientific concepts sprout downwards by means of everyday concepts. Everyday concepts sprout upwards by means of scientific concepts'. Practical activity is considered by Vygotskij and his scholars as the original source of both: only later two identifiable modes of thought, the empirical and the theoretical, are differentatiated. However neither of these can be conceived as fully independent of the other, as both contain latent forms or seeds of the other. The recent developments in the field of embodied cognition (Lakoff and Nuñez,1997; Longo, 1997) emphasise that this is true also for mathematical conceptual system, as it is grounded in our sensorimotor functioning in the world, in our very bodily experience.

The specific project reported in this research study focuses the early approach to mathematical theorems, which characterise the mathematical conceptual system. Following Mariotti et al. (1997), 'a mathematical theorem is meant as the system of three interrelated elements: a statement (i.e. the conjecture produced through experiments and argumentations), a proof (i.e. the special case of argumentation that is accepted by the mathematical community) and a reference theory (including deduction rules – i.e. metatheory – and postulates as well'. Drawing on the analysis of Boero and Garuti (1994) we consider the geometry statements as general, abstract and conditional. The focus on mathematical theorems determines the relevant activities with gears: the construction of reference theories and metatheories, the production of statements and argumentations, the construction and arrangement of proofs.

3. Research Methodology

This study is an example of research for innovation, in which action in the classroom is both a means and a result of progressive knowledge of classroom processes (Bartolini Bussi, 1994). Teachers themselves are members of the research team and share decision-making responsibility together with responsibility for collecting and analysing data of different forms: materials, individual worksheets, transcripts of collective discussions and field notes. Eight teachers with eleven different classes (ranging from the 1st to the 6th grade) have been engaged in the research study since 1996. Because of the differences in pupil age and school situation, the teachers designed a different educational project for each class and these were carried out at the same time. As we are interested in reproducing the conditions for the construction of sense (Brousseau, 1986), a continuous dialectic between the individual projects and the general framework (which offered the shared epistemological analysis) had to be established. Hence there was a shift from the initial fuzzy hypotheses to more precise shared formulations, by means of several interlaced processes. These included the handling of complex mechanisms, reading of historical sources, and solving of complex problems on the one hand, and on the other interpreting and exploiting classroom processes in the most effective way.

4. External time: The chronicle

4.1. *The observed classroom*

We shall mainly discuss data from a 4th grade class in a village school which has been taught by the same two teachers since 1st grade. The teacher in charge of mathematics (Mara Boni) is a teacher-researcher and full member of the research team. The socio-cultural level of the pupils' families is quite low. Initially, the children's language abilities were poor and they received little help, if any, from families with schoolwork: these factors make the classroom an extraordinary laboratory for research in innovation, as it can be assumed that the pupils have developed typical school skills almost exclusively in the classroom, as a result of the intentional work of the teachers. Almost all of the 17 pupils (the class includes one boy with a certified mental disability) are low achievers, not for neurological reasons but because of socio-cultural deprivation. Three pupils receive special socio-psychological support from the local health service. We have chosen one girl, Elisabetta, to guide us in describing the whole process. She is an average pupil with respect to her classmates. Excerpts from her protocols are interspersed in the paper. Two points must be made. First, Elisabetta produced the quoted texts autonomously: the rich verbalisation is the fruit of the patient and systematic work of the teacher, who began in the 1st grade to ask (and help) all the pupils to verbally express their reasoning (by suggesting words) and write it down in full. Second, the overt dialogical style is the result of the interiorisation of the classroom practice of discussion.

4.2. *Prologue*

At the end of the 2nd grade, the whole classroom visited a mill to compare mechanical milling with the hand milling of prehistoric man. All the pupils drew the mill free-hand, with trains of toothed wheels, and described its functioning verbally. In the 3rd grade, under the teacher's guidance, they integrated a new everyday gear, the bicycle cog, in their school experience. At this level, a problem about gears was posed by the teacher, who asked the class to draw a pair of toothed wheels and describe their functioning. In the solution to the problem, most of pupils (including our guide Elisabetta) drew pairs of squared wheels and described their functioning by allowing the pivots to be displaced during motion. We guess that the classroom task of external representation by drawing has no counterpart in the children's everyday experience, which has probably remained at a low perceptual level. This result prompted the teacher to devote considerable time to a dynamic approach to circles, leading to the use of compasses (some details of this process are in Boni, 1997).

4.3. *The roller corrector*

In the 4th grade, the pupils are called back to the topic of gears. The problem concerns a roller corrector, an everyday item of stationery from their schoolbags.

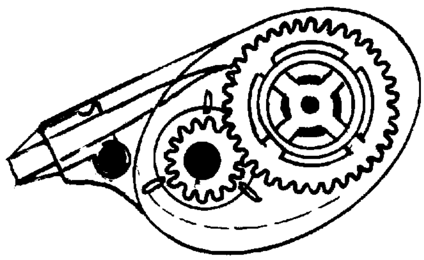


Figure 1. The roller corrector.

Problem A: The roller corrector: Elisabetta's protocol
Look at the roller corrector. How are the wheels made?: size, shape, teeth . . . Think of the wheels in motion: direction, speed, turns . . . How does the roller corrector work?

Wheels size and shape are as follows: there is a big wheel and a small wheel and their size is nearly 1.5 cm and their shape is round. They are positioned obliquely and their teeth are oblique. The direction is as follows: the big one turns anticlockwise and the small one turns clockwise (look at the drawing). Yet, if both turned clockwise or anticlockwise, the teeth might break. [Then she carefully describes the speed and the functioning].

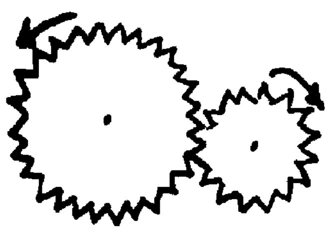


Figure 2. Problem A.

The pair of compasses is used by nearly all the pupils: most of them produce correct statements about the shape and functioning of the tape corrector, picking up all the causal (e.g. *the first wheel pushes the second wheel*) and temporal clauses (e.g. *the wheels turn simultaneously*) that have become part of the speech genre elaborated in the classroom (Bakhtin, in Wertsch, 1991).

4.4. *What about three wheels in gear?*

One month later a new problem is set.

Problem B: What about three wheels in gear? (Elisabetta's protocol)
We have often met planar wheels in pair. What if there were three wheels? How could they be positioned? You can build the possible situations by drawing or by cutting. Remember you must always give the necessary explanations and write down your observations. *1) Wheel n. 1 turns, but we do not know in which direction; let us say that it turns clockwise, then Wheel n. 2 turns anticlockwise, this is sure, and n. 3, how do you think this one turns? I know how: it turns like n.1. Do you know why? Because they*

Figure 3. Problem B.

have to be in gear in the opposite direction. We could do this with fingers too, remember. I've drawn two wheels with arrows in opposite directions. Yet, if we think hard, n. 1 could turn clockwise and n. 2 clockwise too,

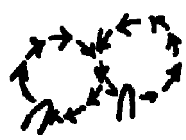


Figure 4. Problem B.

couldn't they? They could not, they would not be in gear. 2) Let us try to draw the wheels in another way. Yes, they are on the same plane; but if I try to turn one we would see that two turn in the same direction and the other turns in the opposite direction. They cannot be in gear, because the first turns clockwise, the second anticlockwise and the third clockwise too, but the first and the third touch each other and so they are not in gear.

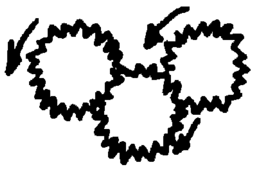


Figure 5. Problem B.

Figure 6. Problem B.

Her statements about motion are correct but the language is not precise: the terms ‘to be in gear’ or ‘to engage’ (Italian: ‘ingranare’) and ‘to turn’ or ‘to work’ (Italian: ‘funzionare’) are used in a random and mistaken manner.

Nearly all the pupils succeed in analysing the three-wheel row with the help of arrows, while only 6 discuss the elementary block. This small number may well be related to the student’s general inability to invent problems by themselves, rather than to a true lack of instruments for solution. Actually, a few days later many pupils spontaneously mention TV advertisements they have seen in which the elementary block is incorrectly drawn as functioning, and comment gravely: *Graphic designers do not know about gears*.

4.5. *What about more than three wheels in gear?*

A generalisation problem is given (Problem C) concerning more than three wheels.

Problem C: What about more than three wheels in gear (Elisabetta’s protocol)

What if there were more than three wheels in gear? After having imagined different situations, try to give some rules. [she analyses the necklace with 4 wheels and says that it works, provided that there is no ‘clover’ configuration inside; then she analyses the necklace with 5 wheels]
These are not like the previous ones: because A, C and E are in gear, but, you see, A and E have the same direction! B and D are in gear easily, D has the same direction as A; but that is close to B; it does not work. [she draws 6 wheels]. *These work: A has the direction of C and E, B the direction of D and F, and you see that they do not block. Now I’ll try with 3 wheels because I’ve already understood one rule. Now I’m really sure I*

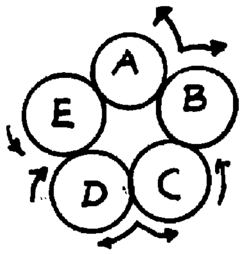


Figure 7. Problem C.

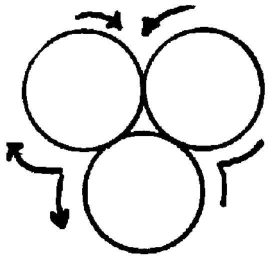


Figure 8. Problem C.

understand. I'm writing the rules.
RULES: 1) The wheels, two by two, if odd are in gear but they block and if they are even they are in gear and do not block [she recalls what she has already said]. *I have done a drawing to make sure.* [She draws 4 and 5-wheels necklaces with alternating colours and with numbers]. *The even ones do not block because even numbers have the same direction and odd numbers the other direction. In the odd one, the even have the same direction and the odd the same direction but 5 has the same direction as 1, as they are odd. This is true for all the gears, the odd one as I said and the even ones as I said.*
Elisabetta tests the syntax of the sign that she has built in the previous problem by applying it to longer and longer necklaces of wheels. She produces and argues statements that are abstract, general and conditional. She also produces a first embryo of reasoning by reductio ad absurdum.

In the classroom only one pupil considers a row of wheels: all the others are interested in finding blocks in necklaces. All the pupils but 2 succeed in finding and interpretting blocks in necklaces with odd numbers of wheels; this is done by means of arrows and using different argumentations. Eleven of the 16 pupils succeed in producing a statement about odd and even necklaces, with different degrees of abstractness, generality and conditionality. The teacher has to rescue 5 pupils (from the lowest level in the class), who have produced some examples but no statement.

4.6. *The balance discussion*

After a short session where the different meanings of some words (e.g. ‘to engage’, ‘to turn’) is clarified, the teacher orchestrates a balance discussion (Bartolini Bussi, 1996). She falls back on the usual strategy of communication, calling on those pupils to be rescued to speak and inviting the others to suggest the words and expressions that are lacking. At the end of the discussion (more than 250 individual interventions) each of the 5 pupils has produced a statement, picking up from the classmates suggestions the words and expressions that they believed best fitted their tacit convictions. A few days later, the pupils stick in their workbooks ample excerpts from the transcripts of the discussion led by the teacher, and these are then read and commented on collectively.

4.7. *The theory*

At this point, the time has come to reorder the activity on the motion of gears, as each pupil has had the opportunity to produce and argue his/her statement. The students are called on to compare their own products with official knowledge. The research team prepares a five-page file on the theory of toothed wheels, where the reference to Heron is explicit (ample excerpts of the text are below). The document is cut into pieces by the teacher and read and commented on collectively, to discover and highlight congruences and differences between the pupils’ and Heron’s text. The teacher sets an additional task: to produce individually the missing or different proofs, according to Heron’s style.

Theory of planar gears
Rows and necklaces of toothed wheels
After more than one year of work we are ready to sum up all that we have learnt. We shall be helped by Heron, an engineer who lived in Egypt in the 1st century A.D. and who studied this problem very well and built many amazing mechanisms. [Definitions to agree on language follow]. *We have tried to build rows and necklaces of toothed wheels on paper (by drawing) and in the mind.* [Several examples follow]. *With many wheels, and much space, time and patience, the rows and the necklaces built on paper or in the mind could be built on the desk too*. [In the discussion the pupils ask to modify this statement by adding a few words, in order to exclude the case of infinitely many wheels, which can be imagined in the mind: the rows and the necklaces of a finite number of wheels]. *With hands, eyes and wheels we have discovered something. To put it better we borrow words from Heron: ‘Two wheels mounted on two axes and in gear with each other by means of teeth turn in opposite directions’. As mathematicians*

have done since ancient times, we shall say that this is a POSTULATE, meaning that it is a safe piece of knowledge for us, one which we are completely sure of. Then we tried to start turning a row of wheels in gear. We worked on paper and in the mind. We tried with a few wheels. [Some examples follow]. *We concluded that: 1) If a row has an even number of wheels, the first one and the last one turn in opposite directions. 2) If a row has an odd number of wheels, the first one and the last one turn in the same direction.*

We PROVE it, borrowing words from Heron: 'If the first wheel turns in the opposite direction of the second (and the second in the opposite direction of the first), the motion of the first is the same as the third. If there is a fourth wheel, we proceed in the same way. Summing up, what happens for three wheels in gear can be repeated for any row with an odd number of wheels, and what happens for two wheels can be repeated for all the rows where wheels are paired two by two, i.e. their number is even'. With this reasoning we can FORESEE the direction of the last wheel of a very long row WITHOUT DOING EXPERIMENTS: it is enough to know whether their number is odd or even. [Then in the same way the problem of necklaces is discussed, up to the block].

5. THE INNER CONTEXT OF PUPILS

In this section we shall consider two crucial issues for the approach to mathematical theorems in the field of kinematics: 1) the genesis and the development of relevant signs for the solution of problems about the motion of gears; 2) the genesis of early theorems about the motion of gears in the case of planar rows and necklaces of wheels. The two processes are strictly interwoven and are treated separately only for analytical purposes.

5.1. *Arrows and other signs*

In all the printed translations of Heron's Mechanics (see for instance Carra de Vaux, 1988) arrows are used. It would be fascinating to think that Heron himself was the author of these arrows, but we know that the figures were redrawn by the French translator: moreover, no arrow is found elsewhere in classical books about kinematics and dynamics until the 20th century, when the geometrical concept of orientation was worked out. Today, from the 1st grade onward, young pupils spontaneously introduce arrows in the graphical representation of gears, probably drawing on out-of-school experience (e.g. comic strips, road signs). Yet this precocious use is only the starting point of the effective use of arrows in solving problems about

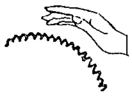
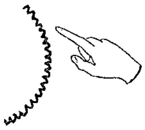
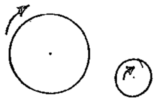
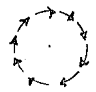
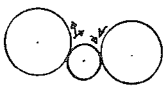
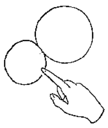
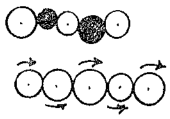
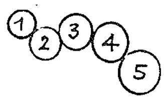
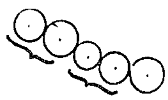
TABLE 1: The genesis of signs				
n	GESTURAL	GRAPHIC	VERBAL	Focus on
1			push this wheel this way this wheel goes this way	global WHEELS
2			this tooth goes this way	pointwise
3			clockwise anticlockwise	global MOTIONS
4			left - right up - down	pointwise
5			wheels turn in pairs	global SYMMETRIC REL. BETWEEN MOTIONS
5'			this wheel pushes that wheel	ASYMMETRIC pointwise / global
6			white-black A-B thumb-index this - the other way	global ALTERNATION
7			one-two-three-four-...	global VARIABLE n ⇩ TO INFINITY
8			wheels are paired	global ARITHMETIC REGULARITY

Table 1. The genesis of signs

gears. In the classrooms we have collected evidence of different statuses for the sign ‘arrow’ and other signs originated by it (see Table I). (1) is the starting point: the arrow represents the gesture of either starting motion or imitating motion. It is a tool for outwardly oriented communication. The sign can be short or long up to embracing the whole wheel as if the hand or the gaze followed one point of the wheel for the whole lap. When problems about two or more wheels are posed, it turns inward, so becoming (2) a tool of semiotic mediation (Vygotskij, 1978). This make it possible to control the direction of motion in the whole wheel and, where necessary, to express the postulate. At the very beginning (1 and 2), attention is still on the physical objects: this is demonstrated by the care with which pupils draw the teeth one by one. Later on, teeth are eliminated (maybe to save time) and circles only are drawn, which although toothless, are still referred to as (toothed) wheels.

Subsequently, two different development paths are observed: one towards a global approach (3) with the use of terms like 'clockwise – anticlockwise'; and another towards a pointwise approach (4) with the use of terms inspired by the body frame of reference, such as 'left-right'; 'upward-downward'. The pointwise approach (the same of Heron) is less functional for the solution of complex problems, as conflicts arise between points-teeth of the same circle-wheel that are symmetrical with respect to the centre (if one goes left the other goes right). Hence the pointwise approach is soon substituted (spontaneously or through social interaction) by a global approach.

Then (5) the relationship between motions come to the foreground. Each pair of 'arrows' establishes a link between two geared wheels. The sign has a gestural counterpart: clasping hands and turning them upward-downward or right-left, to simulate, by means of the symmetrical motion of fists, the symmetrical motion of two wheels in gear. This gesture works for a pair of wheels, but the three wheels situation is puzzling: it is necessary to add a new imaginary element and to follow the motion with the eyes or with the head. Four wheels are too much for everybody.

In order to express the link between two wheels in gear and the alternation of motions, other sign systems are available (6): pair of fingers (thumb – index), colours, letters or arrows. In particular, in long rows (or necklaces) arrows may be drawn alternatively above-below the row (or inside – outside the necklace) to emphasise alternation.

In the pupils' school experience, there is a prime example of alternation: odd – even natural numbers (7). Yet whilst the alternation of colours, letters or fingers still recalls a physical experience with a finite number of wheels (which can be coloured, labelled or touched), the use of numbers may provoke a new reading of problems that is not actually consistent with the original problems about physical gears. This is the shift to infinity, which is perceived as a 'theoretical need', even if meaningless in concrete problems. Actually, the virtual consideration of infinitely many wheels is explicitly requested by pupils to modify the official text that (correctly) considered only gears with a finite number of wheels (see Section 4.7).

Alternation of odd and even numbers with different effects on motion paves the way to producing abstract, general and conditional statements about the variable n. A parity control is stated that makes it possible to foresee or interpret the motion of the nth wheel of a chain. The shift to numbers (7) may well evoke typical arithmetical scripts the pupils have memorized, like the division by two (8). In these final steps the signs that have been developed are connected by a sort of symbolic calculus. This calculus allows to produce results concerning situations that are out of reach of experiments and so offer new insights on the concrete referents.

This is one genesis of the syntax of signs but does not exhaust all the possibilities. For instance the step (5) marks the shift to a symmetrical relationships between (the motions of) two wheels in gear, that is revealed by the use of expressions like *the wheels turn contemporaneously* or *at the same time.* There are pupils who gives (the motions of) the wheels of a pair an asymmetrical function (5'), referring to a causal interpretation: there are *the pushing wheel* and *the pushed wheel,* as if motion transmitted from one to the other. The time order determined by the causal interpretation corresponds to the time order of the exploration of the elements of a long train of wheels, by eyes, by colouring, by arrows (6), by counting (7), in external or inner space.

Evidence of these steps has been collected in all the classrooms, with different individual sequences. In Problem B, Elisabetta reconstructs the gestural meaning of the sign 'arrow' by also referring to fingers: later, in Problems B and C she quickly develops the syntax of the sign, up to Step 7, by showing us a nearly complete and continuous process. Her process is controlled and the strengthening of syntax does not result in semantic loss. For other children we have only fleeting evidence of the use of sign syntax (alternation, usually represented by colours) that can be easily applied to longer and longer rows or necklaces. This rule recalls typical children's games (building necklaces with beads of alternating colours; colouring in patterns drawn on squared paper) and is actually within the reach of low achievers as well.

5.2. *Early theorems about the motion of planar gears*

The tasks we have chosen about the motion of planar gears require production either of previsional hypotheses (How does the gear move?) or interpretative hypothesis (Why does the gear move this way?) without handling the object itself (for the distinction, see Boero and Ferrero, 1994). A priori analysis can be done by transferring the method used by Bergamini (1996) for analysing dynamic experiments in the field of sunshadows. When physical gears are not accessible, experiments may be done in at least three other kinds of space: the external physical space where hands or eyes simulate the motion; the inner space where mental experiments are performed; the external representative space of the paper on which drawings and signs are traced. The choice of space is not limited to a single option: the same pupil can go to and fro between different spaces to solve a complex problem. In all the quoted protocols there is a continuous shift between different spaces: for instance, in one case, the control function is performed with reference to the physical space (fingers in Problem B); in another case, reference is made to a simpler configuration on paper (the elementary block of Problem C). In general, the choice of space is influenced by the complexity of the task. Imaginary wheels can be followed with hands, fingers or eyes in the physical space, when the number of elements is quite small. Working on paper with conventional signs (such as arrows) it is possible to simulate dynamic experiments with many wheels on a static support. But free exploration in the inner space is the only way of capturing cases of impossible motion, because dynamic mental experiments are the only way to cope with them. The inner space is also where the shift to infinity takes place, a process that has no physical counterpart.

Pupils produce early statements about motion considering (Problem B) whether a wheel turns or not and whether a wheel turns clockwise or anticlockwise (left or right). Yet, we are especially interested in a third case: statements concerning the number of wheels in a row or a necklace (Problem C). The hypothesis concerns the set of natural numbers, which is infinite yet partitioned in just two equivalence classes: odd numbers and even numbers. The production of these statements depends on identifying a special regularity in dynamic mental experiments where more and more wheels are added one after another; hence the pupils select cases in a process that advances ‘step by step’. Other models for the production of statements are considered by Boero et al. (1996a; 1996b).

In the protocols, various argumentations are produced to justify individual statements. In the interesting case of the elementary block, we observed at least two different approaches (with possible mixtures): a global approach, like Elisabetta’s one (Section 4.4); a pointwise approach, which is best illustrated by Davide (6th grade).

Problem B: Davide’s Protocol

The first two wheels turn in opposite directions; and this is OK, but there is a third wheel that is in gear with both; it is a kind of block as the teeth should break. Actually the two wheels go in opposite directions and a tooth *would push a tooth of the wheel one way but there is the tooth of the other wheel that pushes this tooth the other way. Conclusion: if the wheels are put in this way they can't turn.*

In both graphical and mental experiments a property drawn from physical experience is used: *'two wheels in gear turn opposite ways'*. For pupils, this does not yet have the status of 'postulate', because there is not yet any theory. Sometimes the pupil is not even able to push the contradiction between the mental experiment and the known property to its extreme consequences and only manages to express a conflict: *I have a wheel with two opposite arrows. I don't understand* (a 3rd grader). Within our framework, only a 'small' step is needed, to shift these argumentations to the status of mathematical proofs i.e. explicitly building the reference theory. Actually, this is brought about by the cultural mediation of the teacher, who introduces the text to be read in the classroom (Section 4.7). Within this theory, pupils' argumentations about the impossible motion assume the status of proofs by reductio ad absurdum. The mental experiments show all their power, as they allow the dynamic exploration of gears that do not actually work and permit production of statements and argumentation for any number of wheels. Actually the motion in the case of the clover configuration (and similar ones) it is not contrasted by the empirical observation that no motion happens, but by the logical argumentation that a contradiction would be generated on the base of what is known.

6. DISCUSSION

This teaching experiment is part of a global project on approaching mathematical theorems that is currently in progress (Mariotti et al., 1997). We have collected evidence that, given a suitable sequence of tasks and proper teacher guidance, most 4th graders can produce general, abstract and conditional statements about motion in the field of experience of gears and take part in the construction of proofs as justifications inside a theory. The data have been confirmed even further in other primary school classrooms and with 6th graders. We claim that the very features of the field of experience of gears (and the effective exploration guided by the teacher) are responsible for this success.

From the outset, the external context is characterised by the presence of concrete referents from everyday life. Yet it is enriched with other elements, namely sign systems that work as tools of semiotic mediation and inhibit the shift to everyday reasoning. The genesis of the signs that we have proposed in the Section 5.1 recalls the genesis of zero in our system of numeracy (Rotman, 1993): moving from physical gears to paper is a shift from a gestural medium (in which physical movements are given ostensibly and transiently in relation to an external apparatus) to a graphic medium (in which permanent signs, originating in these movements, are subject to a syntax given independently of any physical interpretation). On the one hand, the genesis of the signs from gestures allows reconstruction in an elementary case of the link between a mathematical sign and sensor-motor functioning and bodily experience (Lakoff and Núñez, 1997; Longo 1997). We believe the case of gears to be very effective because of the inseparability of machines and gestures in all cultures: in fact, the anthropologist Leroi-Gourham (1943) claims that 'a machine is a device that incorporates not only a tool but also one or more gestures'. On the other hand, signs built in this way feature 'simple' syntax. This puts them within reach of low achievers and makes their introduction a way to 'frame' (Sfard, 1997) the problems, which calls into play metaphors and expectations as to the nature of their meaning. In particular, the shift from alternation of colours to alternation of odd and even numbers suggests the consideration of the variable n on the one hand, and on the other creates the theoretical need to consider infinitely many wheels in the inner space.

Along the way from argumentation to proof, there is no break in so-called 'cognitive unity'. We use this term after Boero et al. (1996b) to mean the continuity between the processes of conjecture production and proof construction, recognisable in the close correspondence between the nature and the objects of the mental activities involved. It is this very continuity that allows the teacher to construct a text concerning proofs that is based on historical sources and that is within the zone of pupils' proximal development, hence immediately and naturally appropriated by them. Such explicit use of history with young pupils is consistent with the Vygotskian claim of the historical construction of consciousness. Pupils become conscious of their own intellectual processes by locating themselves in the history of the self, of their classroom and of their social group. The explicit use of history has already been implemented elsewhere by us and by other researchers to bridge the gap between empirical and theoretical thinking (see Bartolini Bussi, 1996; Boero et al., 1997; Garuti, 1997). In the text used by us, only a limited part of Heron's kinematics, based on a single postulate, is needed for these proofs about the motion of gears (hence we guess it is within the reach of young pupils as well). For this reason we prefer to call this limited theory a 'germ-theory', to emphasise its expansive potency and its tendency to develop into a fully fledged theory (Engestroem, 1987).

7. Concluding Remarks

The introduction of concrete referents into school mathematical activity has been debated for years (Sierpinska, 1995). ‘Realistic mathematics’ (Freudenthal, 1983; Treffers, 1978) and the application of the principle of ‘operative concept formation’ (Bender and Schreiber, 1980) are an expression of a positive attitude. Several reasons are produced to justify the recourse to a ‘real’ context: pupils’ motivation to learn geometry; the need to establish links between school learning and everyday learning; the conceptualisation of mathematics as either ‘a language to describe and interpret reality’ or as ‘a structure that organises reality’. These are all pedagogical, social or philosophical reasons and each can be contrasted with different options. With this exploratory research study we hope to have taken a step ahead, illustrating the cognitive counterpart of activity with everyday concrete referents (in the special case of gears) that allows early approach to theoretical thinking.

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MARIA G. BARTOLINI BUSSI, MARA BONI and FRANCA FERRI
NRSDM Dipartimento di Matematica
Università di Modena e Reggio Emilia
Via G. Campi 213/b
41100 Modena, Italia
E-mail: Bartolini@unimo.it

ROSSELLA GARUTI
IMA CNR
Via De Marini 6
16149 Genova,
Italia